

NEWTON POLYGONS OF L -FUNCTIONS OF POLYNOMIALS

$x^d + ax^{d-1}$ WITH $p \equiv -1 \pmod{d}$

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ABSTRACT. For prime $p \equiv -1 \pmod{d}$ and q a power of p , we obtain the slopes of the q -adic Newton polygons of L -functions of $x^d + ax^{d-1} \in \mathbb{F}_q[x]$ with respect to finite characters χ when p is larger than an explicit bound depending only on d and $\log_p q$. The main tools are Dwork's trace formula and Zhu's rigid transform theorem.

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1. MAIN RESULTS

Let $q = p^h$ be a power of the rational prime number p . Let v be the normalized valuation on $\overline{\mathbb{Q}}_p$ with $v(p) = 1$. For a polynomial $f(x) \in \mathbb{F}_q[x]$, let $\hat{f} \in \mathbb{Z}_q[x]$ be its Teichmüller lifting. For a finite character $\chi : \mathbb{Z}_p \rightarrow \mathbb{C}_p^\times$ of order p^{m_x} , define the L -function

$$L^*(f, \chi, t) = \exp \left(\sum_{m=1}^{\infty} S_m^*(f, \chi) \frac{t^m}{m} \right), \quad (1)$$

where $S_m^*(f, \chi)$ is the exponential sum

$$S_m^*(f, \chi) = \sum_{x \in \mu_{q^m-1}} \chi(\text{Tr}_{\mathbb{Q}_{q^m}/\mathbb{Q}_p} \hat{f}(x)) \quad (2)$$

and μ_n is the group of n -th roots of unity. Then $L^*(f, \chi, t)$ is a polynomial of degree $p^{m_x-1}d$ by Adolphson-Sperber [AS87] and Liu-Wei [LW07]. We denote $\text{NP}_q(f, \chi, t)$ the q -adic Newton polygon of $L^*(f, \chi, t)$.

We fix a character $\Psi_1 : \mathbb{Z}_p \rightarrow \mathbb{C}_p^\times$ of order p , and denote $L^*(f, t) = L^*(f, \Psi_1, t)$ and $\text{NP}_q(f, t) = \text{NP}_q(f, \Psi_1, t)$. When $p \equiv 1 \pmod{d}$, it is well-known that $\text{NP}_q(f, t)$ coincides the Hodge polygon with slopes $\{i/d : 0 \leq i \leq d-1\}$.

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Let a be a nonzero element in \mathbb{F}_q . For $f(x) = x^d + ax^s$ ($s < d$), Liu-Niu and Zhu obtained the slopes of $\text{NP}_q(f, t)$ for p large enough under certain conditions in [LN14, Theorem 1.10] and [Zhu14], but these conditions are not so easy to check. For $f(x) = x^d + ax$, Zhu, Liu-Niu and Ouyang-J. Yang obtained the slopes in [Zhu14, Theorem 1.1], [LN11, Theorem 1.10] and [OY16, Theorem 1.1], see also R. Yang [Yan03, §1 Theorem] for earlier results.

In [DWX16], Davis–Wan–Xiao gave a result on the behavior of the slopes of $\text{NP}_q(f, \chi, t)$ when the order of χ is large enough. In this way for p sufficiently large, they can obtain the slopes of $\text{NP}_q(f, \chi, t)$ based on the slopes of $\text{NP}_q(f, \chi_0, t)$ with χ_0 a character of order p^2 . In [Niu11], Niu gave a lower bound of the Newton polygon $\text{NP}_q(f, \chi, t)$. In [OY16, Theorem 4.3], Ouyang–Yang showed that if the Newton polygon of $L^*(f, t)$ is sufficiently close to its Hodge polygon, the slopes of $\text{NP}_q(f, \chi, t)$ for χ in general follow from the slopes of $\text{NP}_q(f, t)$. As a consequence they obtained the slopes of $\text{NP}_q(x^d + ax, \chi, t)$ when p is bigger than an explicit bound depending only on d and h .

Our main results are the following two theorems.

Theorem 1. *Let $f(x) = x^d + ax^{d-1}$ be a polynomial in $\mathbb{F}_q[x]$ with $a \neq 0$. Let $N(d) = \frac{d^2+3}{4}$ for $q = p$ and $\frac{d^2}{2}$ for general q . If $p \equiv -1 \pmod{d}$ and $p > N(d)$, the q -adic Newton polygon of $L^*(f, t)$ has slopes*

$$\{w_0, w_1, \dots, w_{d-1}\},$$

where

$$w_i = \begin{cases} \frac{(p+1)i}{d(p-1)}, & \text{if } i < \frac{d}{2}; \\ \frac{(p+1)i-d}{d(p-1)} = \frac{1}{2}, & \text{if } i = \frac{d}{2}; \\ \frac{(p+1)i-2d}{d(p-1)}, & \text{if } i > \frac{d}{2}. \end{cases}$$

Remark. (1) For general p , write $pi = dk_i + r_i$ with $1 \leq i, r_i \leq d-1$. If $r_i > s$ for any $1 \leq i \leq s$, then one can decide that the first $s+1$ slopes of $\text{NP}_q(f, t)$ are $\{0, \frac{k_1+1}{p-1}, \dots, \frac{k_s+1}{p-1}\}$ by our method for sufficiently large p . For the rest of slopes, one needs to calculate the determinants of submatrices of ‘‘Vandermonde style’’ matrices.

(2) The slopes in our case coincide Zhu’s result in [Zhu14].

Theorem 2. *Assume $f(x)$ and $N(d)$ as above. For any non-trivial finite character χ , if $p \equiv -1 \pmod{d}$ and $p > \max\{N(d), \frac{h(d^2-1)}{4d} + 1\}$, the q -adic Newton polygon of $L^*(f, \chi, t)$ has slopes*

$$\{p^{1-m_\times}(i + w_j) : 0 \leq i \leq p^{m_\times-1} - 1, 0 \leq j \leq d-1\}.$$

2. PRELIMINARIES

2.1. Dwork’s trace formula. We will recall Dwork’s work for $f(x) = x^d + ax^{d-1}$. For general f , one can see [OY16, §2].

Let $\gamma \in \mathbb{Q}_p(\mu_p)$ be a root of the Artin-Hasse exponential series

$$E(t) = \exp\left(\sum_{m=0}^{\infty} p^{-m} t^{p^m}\right)$$

such that $v(\gamma) = \frac{1}{p-1}$. Fix a $\gamma^{1/d} \in \bar{\mathbb{Q}}_p$. Let

$$\theta(t) = E(\gamma t) = \sum_{m=0}^{\infty} \gamma_m t^m$$

be Dwork's splitting function. Then $v(\gamma_m) \geq m/(p-1)$, and $\gamma_m = \gamma^m/m!$ for $0 \leq m \leq p-1$. Let

$$F(x) = \theta(x^d)\theta(ax^{d-1}) = \sum_{i=0}^{\infty} F_i x^i,$$

then

$$F_i = \sum_{dm+(d-1)n=i} \gamma_m \gamma_n a^n.$$

One can see $m+n \geq i/d$ and $v(F_i) \geq \frac{i}{d(p-1)}$.

Set $A_1 = (F_{pi-j}\gamma^{(j-i)/d})_{i,j \geq 0}$. This is a nuclear matrix over $\mathbb{Q}_q(\gamma^{1/d})$ with

$$v(F_{pi-j}\gamma^{(j-i)/d}) \geq \frac{pi-j}{d(p-1)} + \frac{j-i}{d(p-1)} = \frac{i}{d}.$$

We extend the Frobenius φ to $\mathbb{Q}_q(\gamma^{1/d})$ with $\varphi(\gamma^{1/d}) = \gamma^{1/d}$.

Theorem 3 (Dwork). *Let $A_h = A_1\varphi(A_1)\cdots\varphi^{h-1}(A_1)$. Then*

$$L^*(f, t) = \frac{\det^{\varphi^{-1}}(I - tA_h)}{\det^{\varphi^{-1}}(I - tqA_h)}.$$

2.2. Zhu's rigid transformation theorem. Let $U_1 = (u_{ij})_{i,j \geq 0}$ be a nuclear matrix over $\mathbb{Q}_q(\gamma^{1/d})$. Then the Fredholm determinant $\det(I - tU_1)$ is well defined and p -adic entire (see [Ser62]). Write

$$\det(I - tU_1) = c_0 + c_1 t + c_2 t^2 + \cdots.$$

For $0 \leq t_1 < t_2 < \cdots < t_s$, denote by $U_1(t_1, \dots, t_s)$ the principal sub-matrix consisting of (t_i, t_j) -entries of U_1 for $1 \leq i, j \leq s$. In particular, denote $U_1[s] = U_1(0, 1, \dots, s-1)$. Then we have $c_0 = 1$ and for $s \geq 1$,

$$c_s = (-1)^s \sum_{0 \leq t_1 < t_2 < \cdots < t_s} \det U_1(t_1, t_2, \dots, t_s).$$

Let $U_h = U_1\varphi(U_1)\cdots\varphi^{h-1}(U_1)$. Write

$$\det(I - tU_h) = C_0 + C_1 t + C_2 t^2 + \cdots.$$

Theorem 4. (See [Zhu12, Theorem 5.3].) *Suppose $(\beta_s)_{s \geq 0}$ is a strictly increasing sequence such that*

$$\beta_i \leq v(a_{ij}) \text{ and } \lim_{s \rightarrow +\infty} \beta_s = +\infty.$$

If

$$\sum_{s < i} \beta_s \leq v(\det U_1[i]) \leq \frac{\beta_i - \beta_{i-1}}{2} + \sum_{s < i} \beta_s$$

holds for every $1 \leq i \leq k$, then $v(C_i) = hv(\det U_1[i])$ for $1 \leq i \leq k$ and

$$\text{NP}_q(\det(I - tA_h[k])) = \text{NP}_p(\det(I - tA_1[k])).$$

3. SLOPES OF THE NEWTON POLYGON OF $L^*(f, \chi, t)$

From now on, we assume $p \equiv -1 \pmod{d}$ and write $p = dk - 1$.

3.1. The case $\chi = \Psi_1$.

Lemma 5. *Let $M(s) = (a_{ij})_{1 \leq i, j \leq s}$ be an $s \times s$ matrix with entries*

$$a_{i,j} = \frac{a^{i+j}}{(ki - i - j)!(i + j)!}.$$

Then $v(\det M(s)) = 0$ for $1 \leq s \leq d - 1$.

Proof. Denote $x[0] = 1$ and $x[n] := x(x-1) \cdots (x-n+1)$ for $n \geq 1$. Then $x[n]$ is a polynomial of x of degree n and $\{(x+j)[t] : 0 \leq t \leq j-1\}$ is a basis of the space of polynomials of degree $\leq j-1$. Thus we can write

$$((k-1)x-1)[j-1] = c_0(j) + \sum_{t=1}^{j-1} c_t(j) \cdot (x+j)[t].$$

Let $x = -j$, we get

$$c_0(j) = ((k-1)(-j)-1)[j-1] = ((1-k)j-1)[j-1].$$

For any $1 \leq u \leq j-1$,

$$1 \leq (k-1)j + u < kj \leq k(d-1) \leq p.$$

Hence $p \nmid (1-k)j - u$ and $v(c_0(j)) = 0$.

Let $D = \text{diag}\{a, a^2, \dots, a^s\}$ and $M' = (a'_{ij})_{1 \leq i, j \leq s}$ with $a'_{ij} = a_{ij}a^{-i-j}$, then

$$M(s) = DM'D. \quad (3)$$

Let $a''_{ij} := (ki - i - 1)!(i + s)!a'_{ij}$. Then

$$\begin{aligned} a''_{ij} &= (ki - i - 1)[j-1] \cdot (i + s)[s-j] \\ &= \sum_{t=0}^{j-1} c_t(j) \cdot (i + j)[t] \cdot (i + s)[s-j], \\ &= \sum_{t=0}^{j-1} c_t(j) \cdot (i + s)[s-j+t] \\ &= \sum_{t=1}^j (i + s)[s-t] \cdot c_{j-t}(j). \end{aligned}$$

Define $c_{j-t}(j) := 0$ for $j < t$. Write $M'' = (a''_{ij})_{1 \leq i, j \leq s}$, $M_1 = ((i + s)[s-t])_{1 \leq i, t \leq s}$ and $M_2 = (c_{j-t}(j))_{1 \leq t, j \leq s}$. Then

$$M'' = M_1 M_2. \quad (4)$$

Write

$$x[n] = \sum_{t=0}^n c'_t(n) x^t,$$

then $c'_n(n) = 1$ and

$$(i + s)[s-j] = \sum_{t=0}^{s-j} c'_t(s-j)(i + s)^t.$$

Define $c'_t(n) := 0$ for $t > n$. Write $M_{11} = ((i + s)^{t-1})_{1 \leq i, t \leq s}$ and $M_{12} = (c'_{t-1}(s-j))_{1 \leq t, j \leq s}$. Then

$$M_1 = M_{11} M_{12}. \quad (5)$$

Notice that M_{11} is a Vandermonde matrix with determinant $\det M_{11} = \prod_{t=1}^s t^{s-t}$. One can also easily find

$$\det M_{12} = (-1)^{\lfloor s/2 \rfloor} \quad \text{and} \quad \det M_2 = \prod_{i=1}^s c_0(i). \quad (6)$$

Now by (3), (4), (5) and (6),

$$\det M(s) = a^{s(s+1)} (-1)^{\lfloor s/2 \rfloor} \prod_{i=1}^s \frac{i^{s-i} c_0(i)}{(ki - i - 1)!(i + s)!}.$$

Hence $v(\det M(s)) = 0$. □

Denote $O(x)$ a number in $\overline{\mathbb{Q}_p}$ with valuation $\geq v(x)$ for $x \in \overline{\mathbb{Q}_p}$.

Lemma 6. (i) For $i + j < d$, $F_{pi-j} = \gamma^{ki} (a_{ij} + O(\gamma))$.

(ii) For $i + j \geq d$, $v(F_{pi-j}) = ki - 1$ and

$$F_{pi-(d-i)} = \frac{\gamma^{ki-1} (1 + O(\gamma))}{(ki - 1)!}.$$

Proof. Let

$$m = \begin{cases} ki - i - j, & \text{if } j < d - i; \\ ki - i - j + d - 1, & \text{if } j \geq d - i, \end{cases}$$

$$n = \begin{cases} i + j, & \text{if } j < d - i; \\ i + j - d, & \text{if } j \geq d - i. \end{cases}$$

Then $pi - j = dm + (d - 1)n$ and $0 \leq n \leq d - 1$. This lemma follows from

$$F_{pi-j} = \sum_{l \geq 0} \gamma_{m-(d-1)l} \gamma_{n+dl} a^{n+dl} = \gamma_m \gamma_n a^n (1 + O(\gamma)) = \frac{\gamma^{m+n} a^n}{m!n!} (1 + O(\gamma)). \quad \square$$

Proposition 7. For $1 \leq s \leq d - 1$, the valuation of $\det A_1[s + 1]$ is $w_0 + w_1 + \dots + w_s$.

Proof. Note that the first row of A_1 is $(1, 0, 0, \dots)$. Let A be the matrix by deleting the first row and column of $A_1[s + 1]$. Then $\det A_1[s + 1] = \det A$.

Let $D_1 = \text{diag}\{\gamma^{0/d}, \gamma^{1/d}, \dots, \gamma^{s/d}\}$, $D_2 = \text{diag}\{\gamma^{k-1}, \gamma^{2k-1}, \dots, \gamma^{(d-1)k-1}\}$ and $B[s] = (\gamma^{1-ki} F_{pi-j})_{1 \leq i, j \leq s}$. Then $A = D_1^{-1} D_2 B[s] D_1$. It suffices to compute $v(\det B[s])$.

Note that for $s = d - 1$,

$$B[d - 1] = \begin{pmatrix} \gamma a_{11} + O(\gamma^2) & \cdots & \gamma a_{1,d-2} + O(\gamma^2) & \frac{1+O(\gamma)}{(k-1)!} \\ \vdots & \ddots & \frac{1+O(\gamma)}{(2k-1)!} & b_{2,d-1} \\ \gamma a_{d-2,1} + O(\gamma^2) & \cdots & \cdots & \vdots \\ \frac{1+O(\gamma)}{((d-1)k-1)!} & b_{d-1,2} & \cdots & b_{d-1,d-1} \end{pmatrix}$$

with $v(b_{ij}) = 0$. If $1 \leq s \leq \frac{d-1}{2}$, then

$$B[s] = \begin{pmatrix} \gamma a_{11} + O(\gamma^2) & \cdots & \gamma a_{s1} + O(\gamma^2) \\ \vdots & \ddots & \vdots \\ \gamma a_{s1} + O(\gamma^2) & \cdots & \gamma a_{ss} + O(\gamma^2) \end{pmatrix}$$

has determinant

$$\det B[s] = \gamma^s (\det M(s) + O(\gamma)).$$

The valuation of $\det B[s]$ is $sv(\gamma)$.

If $\frac{d}{2} \leq s \leq d-1$, then

$$B[s] = \begin{pmatrix} B[d-1-s] & P_1 \\ & P_2 \\ & & Q \end{pmatrix}.$$

The valuation of any entry of $B[d-1-s]$, P_1, P_2 is $v(\gamma)$ and

$$Q \equiv \begin{pmatrix} 0 & \cdots & \frac{1}{(k-1)!} \\ \frac{1}{((d-1)k-1)!} & & * \end{pmatrix} \pmod{\gamma}.$$

Thus Q is invertible over the ring of integers of $\mathbb{Q}_p(\gamma)$. The determinant

$$\det B[s] = \det Q \det(B[d-1-s] - P_1 Q^{-1} P_2) = \det Q \det B[d-1-s](1 + O(\gamma))$$

has valuation $(d-1-s)v(\gamma)$.

Finally, $A = D_1^{-1} D_2 B[s] D_1$ has valuation

$$\left(\sum_{i=1}^s (ki-1) + \min\{s, d-1-s\} \right) v(\gamma) = w_0 + w_1 + \cdots + w_s. \quad \square$$

Proof of Theorem 1. For $1 \leq s \leq d-1$, we have

$$\begin{aligned} v(\det A_1[s+1]) &= \sum_{i \leq s} w_i \\ &= \begin{cases} \frac{s(s+1)}{2d} + \frac{s(s+1)}{d(p-1)}, & \text{if } s \leq (d-1)/2; \\ \frac{s(s+1)}{2d} + \frac{(d-s)(d-s-1)}{d(p-1)}, & \text{if } s \geq d/2; \end{cases} \\ &\leq \frac{s(s+1)}{2d} + \frac{d^2-1}{4d(p-1)}. \end{aligned}$$

If $p > \frac{d^2+3}{4}$, then $\frac{d^2-1}{4d(p-1)} < 1/d$. For $0 \leq t_0 < t_1 < \cdots < t_s$, assume $t_s \neq s$. Since

$$v(F_{pi-j} \gamma^{(j-i)/d}) \geq i/d,$$

we have

$$v(\det A_1[t_0, \dots, t_s]) \geq \frac{s^2 + s + 2}{2d} > v(\det A_1[s+1]).$$

Thus $v(c_{s+1}) = v(\det A_1[s+1]) = \sum_{i \leq s} w_s$ and $\{w_0, w_1, \dots, w_{d-1}\}$ are slopes of $\text{NP}_p(\det(I - tA_1))$.

If moreover $p > \frac{d^2}{2}$, then $p \geq \frac{d^2+1}{2}$ and $\frac{d^2-1}{4d(p-1)} \leq \frac{1}{2d}$. Choose $\beta_i = i/d$ in Theorem 4, we have

$$v(C_{s+1}) = h(w_0 + w_1 + \cdots + w_s)$$

and

$$\text{NP}_q(\det(I - tA_h[d])) = \text{NP}_p(\det(I - tA_1[d])).$$

Thus w_0, w_1, \dots, w_{d-1} are q -adic slopes of $\text{NP}_q(\det^{\varphi^{-1}}(I - tA_h))$.

By Theorem 3,

$$\det^{\varphi^{-1}}(I - tA_h) = L^*(f, t) \det^{\varphi^{-1}}(I - tqA_h).$$

Since the valuation of any entry of A_h is ≥ 0 , the q -adic slopes of $\det^{\varphi^{-1}}(I - tA_h)$ are ≥ 0 and the q -adic slopes of $\det^{\varphi^{-1}}(I - tqA_h)$ are ≥ 1 . Thus any q -adic slope of $\det^{\varphi^{-1}}(I - tA_h)$ less than 1 must be a q -adic slope of $L^*(f, t)$. But $L^*(f, t)$ has degree d , hence w_0, \dots, w_{d-1} are all slopes of $L^*(f, t)$. \square

3.2. The case for general χ . Let $f(x) \in \mathbb{F}_q[x]$ be a polynomial with degree d . Assume $p \nmid d$. Let $\text{NP}(f, x)$ be the piecewise linear function whose graph is the q -adic Newton polygon of $\det(I - tA_h)$. Let $\text{HP}(f, x)$ be the piecewise linear function whose graph is the polygon with vertices

$$\left(k, \frac{k(k-1)}{2d}\right), \quad k = 0, 1, 2, \dots$$

Then $\text{NP}(f, x) \geq \text{HP}(f, x)$ (cf. [LW07, OY16]). Set

$$\text{gap}(f) = \max_{x \geq 0} \{\text{NP}(f, x) - \text{HP}(f, x)\}.$$

Theorem 8. (See [OY16, Theorem 4.3].) *Let $0 = \alpha_0 < \alpha_1 < \dots < \alpha_{d-1} < 1$ denote the slopes of the q -adic Newton polygon of $L^*(f, t)$. If $\text{gap}(f) < 1/h$, then the q -adic Newton polygon of $L^*(f, \chi, t)$ has slopes*

$$\{p^{1-m_x}(i + \alpha_j) : 0 \leq i \leq p^{m_x-1} - 1, 0 \leq j \leq d-1\}$$

for any non-trivial finite character χ .

Proof of Theorem 2. The slopes of $\text{NP}(f, x)$ are

$$\{i + w_j : i \geq 0, 0 \leq j \leq d-1\}.$$

Notice that

$$\sum_{i=0}^{d-1} w_i = \sum_{i=0}^{d-1} \frac{i}{d},$$

$\text{NP}(f, x) - \text{HP}(f, x)$ is a periodic function with period d . For $0 \leq k < d$,

$$\text{NP}(f, x) - \text{HP}(f, x) \leq \sum_{i \leq (d-1)/2} \frac{2i}{d(p-1)} \leq \frac{d^2-1}{4d(p-1)}.$$

If $p > \frac{h(d^2-1)}{4d} + 1$, then $\text{gap}(f) < 1/h$ and this concludes the proof. \square

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